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Enumeration of simple random walks and tridiagonal matrices

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Abstract

We present some old and new results in the enumeration of random walks in one dimension, mostly developed in work on enumerative combinatorics. The relation between the trace of the nth power of a tridiagonal matrix and the enumeration of weighted paths of n steps allows an easier combinatorial enumeration of paths. It also seems promising for the theory of tridiagonal random matrices.

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1. Introduction

Already at the foundation of the theory of random matrices by Wigner [1] the relevance of the combinatorics of random walks was recognized. The following decades witnessed an explosion in the theory and applications of random matrices. A number of specific techniques were devised and the relation with the combinatorics of random walks was almost forgotten, with a few remarkable exceptions [2]. An interesting and recently investigated open problem [3–8] is the spectral density of non-Hermitian tridiagonal random matrices. Here the enumeration of one-dimensional random walks, where an independent random variable is associated with each step, plays a dominant role [11]. Since the subject of random walks is very basic and useful, it has been thoroughly studied both by mathematicians and physicists for a long time, yet with techniques and aims so different that the relevant literature about inhomogeneous walks from one group ignores that of the other group. The aim of this paper is to review some old and recent results in enumerative combinatorics of random walks pertinent to the analysis of tridiagonal matrices in a language accessible to physicists, with examples, comments and relations that are not easily available in the literature.

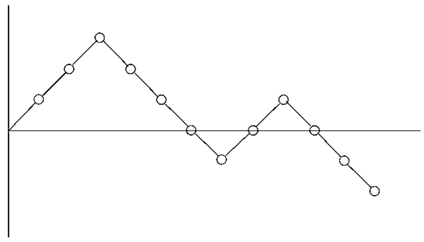


Figure 1. The vertical axis shows the sites visited by a random walker versus discrete time.

A simple walk of p steps on the lattice of integers may be coded by a sequence $\{\mu_1, \ldots, \mu_p\}$, where each μ_k may take the values 1 (right move) or -1 (left move). The walk is simple by the fact that we exclude the move $\mu_k = 0$. The index k can be regarded as discrete time. Since our walks start from the origin, at time k the walker is on site $s(k) = \mu_1 + \cdots + \mu_k$. It is very convenient to view the walk as a continuous broken line in the lattice (k, s) of time and site occupancy. Most of the terminology is based on this picture. An illustration of a simple walk which we consider is given in figure 1; the discrete time that numbers the steps is measured on the horizontal axis, and the positions of the walker on the line are recorded on the vertical axis.

A little terminology is useful. Simple walks, where the end site is the origin, that is $s(p) = 0$, are 'closed'. A simple closed walk is 'weakly positive' if $s(k) \geqslant 0$ for all k. If we further require that it returns to the origin only at the end, the simple walk is 'strictly positive'. Weakly positive closed walks are also known as Dyck paths. The width of a walk is the number of different visited sites.

Simple walks can be classified and then counted according to different properties. A large number of results in the enumeration of simple walks have been obtained by a method which we call 'iteration by length': the counting of walks with a given property is related to the counting of similar but shorter ones, attached sequentially. In terms of generating functions, the method leads to a system of equations which, in several important cases, can be solved explicitly. One of the best known examples is the enumeration of returns to the origin in closed positive walks. In section 2 we illustrate the method by evaluating the number of peaks and valleys in several ensembles of walks. It is a very simple exercise given the known methods in enumerative combinatorics [12–14], and the method is familiar to physicists as it recalls a Dyson equation in quantum field theory.

Recently we developed a technique [11] to count the number of visits of each site for an ensemble of simple walks. The method is again recursive but here properties of longer and wider walks are computed in terms of shorter and narrower ones. We call this method 'iteration by top insertions'. It leads to explicit results in the form of sums of products of binomial coefficients, the sum being actually in several indices taking values in the compositions of an integer, occasionally with further restrictions. Since the results are complicated, techniques to simplify them would be very useful. We think that this method is promising for the analysis of the spectrum of tridiagonal random matrices and the related random hopping problem in one dimension, where few exact results are available, especially in the non-Hermitian case. In this introduction we shall describe in detail the connection of this counting problem for simple walks with tridiagonal matrices.

We then found that our method 'iteration by top insertions' had already been discovered in the form of enumeration of vertices of planar rooted trees [13, 21, 22]. We present this relation in section 3.

Occasionally a problem may be analysed both by the simpler method of section 2 as well as by the more complex one of section 3. We provide an example by evaluating the area under Dyck paths, recently discussed by Jonsson and Wheater [23], with the simple method. While our discussion does not add to their solution, the interpretation in terms of a tridiagonal matrix facilitates its determination.

The method in section 2 is most useful for homogeneous (or almost homogeneous) random walks, where all steps (or almost all of them) have the same probability. Some early papers are [15, 16]. We remark that even the completely inhomogeneous walks, where each step is associated with an independent random variable, were enumerated [17] by methods similar to the one we describe in section 2.

As is often the case, computing the same quantities in two different ways produces identities which could be hard to prove in a direct way. For example, the sums over compositions of an integer of products of binomials, introduced in section 3, will be shown to yield simple trigonometric sums. An early example of this type of sum is provided by equation (22) in Klarner [18].

Most of the methods described in this paper have been developed for walks where the move $\mu_k = 0$, that is an horizontal step, is also allowed. Dyck paths properly generalized to include such moves are called Motzkin paths. Enumeration formulae become more complicated and, for the introductory purpose of this paper, we avoid these walks. Some results for generating functions for Dyck and Motzkin paths and their relation to continued fractions were recently summarized by Krattenthaler [19].

Ensembles of walks are among the most basic topics in probability theory and their statistical properties are relevant in a large class of models in statistical mechanics.

In one dimension, to each step e_k of a walk γ one may assign a weight $w(e_k)$. The whole path is then attributed a weight $w(\gamma) = w(e_1)w(e_2)\dots w(e_p)$. Finally one is interested in summing over walks in some set Γ, weighted with their own factor $w(\gamma)$. One encounters this procedure in the discrete approximation of functional integrals $\int \mathcal{D}x\ f[x(t)]$, where the integration over continuous positive functions $x(t)$ with boundary conditions $x(t_0) = 0 = x(t_f)$ is replaced by an ensemble of Dyck paths. The enumeration methods described here may be useful for the evaluation of functional integrals where the paths may have various restrictions. In example 3 of section 3, a few cases of weighted sums of Dyck paths are provided.

1.1. Random walks and tridiagonal matrices

There is a very close relation between random walks and tridiagonal matrices, and the latter are known to be a powerful tool for counting walks with various specifications. Tridiagonal matrices are frequently used to describe the motion of a particle in a one-dimensional chain.

Since the walks we are considering are made of steps ± 1, we actually consider bidiagonal matrices of size $N \times N$:

$$M(\boldsymbol{a}, \boldsymbol{b}) = \begin{pmatrix} 0 & b_1 & & & & \\ a_1 & 0 & b_2 & & & \\ & a_2 & 0 & \dots & & \\ & & & & & \\ & & & & & b_{N-1} \\ & & & & a_{N-1} & 0 \end{pmatrix}. \tag{1.1}$$

The starting point of our discussion is the fact that the explicit expansion of $M(\boldsymbol{a}, \boldsymbol{b})^p_{i,j}$ immediately suggests ordering the various terms in the sum according to some property of the random walk that connects site i to site j in p steps.

In more detail, if we identify the matrix element $M(\boldsymbol{a}, \boldsymbol{b})_{ij}$ with the symbol (i, j), and agree that repeated neighbouring indices are summed, we have: $(M^p)_{ij} = (i, i_1)(i_1, i_2)\ldots(i_{p-1}, j)$. Since the matrix is bidiagonal, either $i_{k+1} = i_k - 1$ or $i_{k+1} = i_k + 1$. Because of this, the pair (i_k, i_{k+1}) can also be identified with a move from site i_k to a neighbouring site i_{k+1} in a simple walk. Therefore, each sequence of p factors, $(i, i_1)\ldots(i_{p-1}, j)$, not only represents a product of matrix elements but also a random walk of p steps that connects site i to site j. The corresponding product of matrix elements may be interpreted as the 'weight' of the walk.

In this picture, the evaluation of $M(\boldsymbol{a}, \boldsymbol{b})^p_{ij}$ consists in summing the weights of all walks of p steps from i to j. If we sum over lengths we get the resolvent, which is the generating function for all weighted simple paths from i to j:

$$\begin{aligned} F(z, \boldsymbol{a}, \boldsymbol{b}; i, j) &= \sum_p z^p M(\boldsymbol{a}, \boldsymbol{b})^p_{i,j} = [I - zM(\boldsymbol{a}, \boldsymbol{b})]^{-1}_{i,j} \\ &= (-1)^{i+j} \frac{\det[I - zM(\boldsymbol{a}, \boldsymbol{b}); j, i]}{\det[I - zM(\boldsymbol{a}, \boldsymbol{b})]}. \end{aligned} \tag{1.2}$$

Here $[A; i, j]$ is the matrix A with row i and column j removed. This important result is very well known (see, for instance, theorem 4.7.2 in [14]).

It is very useful to note that the matrix $M(\boldsymbol{a}, \boldsymbol{b})$ is similar to the matrix $M(\mathbf{1}, \boldsymbol{x})$ where $x_i = a_i b_i$:

$$M(\mathbf{1}, \boldsymbol{x}) = \begin{pmatrix} 0 & x_1 & & & & \\ 1 & 0 & x_2 & & & \\ & 1 & 0 & \cdots & & \\ & & & & & \\ & & & & & x_{N-1} \\ & & & & 1 & 0 \end{pmatrix}. \tag{1.3}$$

In the relation $M(\boldsymbol{a}, \boldsymbol{b}) = S\,M(\mathbf{1}, \boldsymbol{x})\,S^{-1}$ the matrix S is diagonal, with entries $s_1 = 1$, $s_2 = a_1, \ldots, s_N = a_1 a_2 \ldots a_{N-1}$. Therefore

$$[M(\boldsymbol{a}, \boldsymbol{b})^p]_{ij} = \frac{s_i}{s_j}[M(\mathbf{1}, \boldsymbol{x})^p]_{ij}. \tag{1.4}$$

From this point, we exploit the similarity (1.4), where we note that the factor s_i/s_j does not depend on p. To consider the matrix $M(\mathbf{1}, \boldsymbol{x})$, as we shall do hereafter, is a great simplification. In each walk, only upward steps $(k, k+1)$ correspond to nontrivial factors x_k. Several walks may have the same weight since they contain, but with different order, the same intermediate steps $(k, k+1)$. By collecting these equal contributions, we have a useful expression in terms of certain counting numbers of random walks. The simplest example is

$$[M(\mathbf{1}, \boldsymbol{x})^{2p}]_{1,1} = \sum_{t=1}^{N-1} \sum_{\kappa(p,t)} N(n_1, \ldots, n_t) x_1^{n_1} x_2^{n_2} \ldots x_t^{n_t}. \tag{1.5}$$

Here $N(n_1, \ldots, n_t)$ is the number of Dyck paths that make n_j upward steps $(j-1, j)$ from site $j-1$ to site j, $j = 1 \ldots t$. Their length is $2p = 2(n_1 + \cdots + n_t)$ and t is the height, which is bounded by the size of the matrix.

The multiple sums in $n_1, \ldots, n_t \geqslant 1$ such that $n_1 + \cdots + n_t = p$ are summarized as the sum on the compositions of p into t integers. We denote the set of compositions of p into t integers as $\kappa(p; t)$.

This formula gives the explicit representation of $M(\mathbf{1}, \boldsymbol{x})^{2p}_{11}$ as a *polynomial* in the entries of the matrix.

At the end of section 3 we describe the general case $[M(\mathbf{1}, \boldsymbol{x})^p]_{i,j}$ which corresponds to the counting of random paths of p steps from site i to site j, without positivity restriction.

In the case $\boldsymbol{x} = \mathbf{1}$ each walk has unit weight and one obtains the counting number of simple walks from site i to site j of length p, restricted in the lattice of points from 1 to N, in terms of a simple matrix quantity. Since eigenvalues and eigenvectors are known, the multiple sums of the counting numbers in equation (3.49) are equal to the trigonometric sum:

$$[M(\mathbf{1},\mathbf{1})^p]_{i,j} = \frac{2^{p+1}}{N+1}\sum_{k=1}^{N}\cos^p\left(\frac{k\pi}{N+1}\right)\sin\left(\mathrm{i}\frac{k\pi}{N+1}\right)\sin\left(\mathrm{j}\frac{k\pi}{N+1}\right). \tag{1.6}$$

One can use random bidiagonal matrices to count random walks with restrictions. The trick is to assign the variables x_i a probability density such that the contribution of unwanted walks is made to vanish on the average.

For example, in [11] we studied the 'q root of unity' matrix ensemble, generalizing a problem first suggested by Zee [5]. A random matrix $M(\mathbf{1},\boldsymbol{x})$ in the ensemble is characterized by independent random complex entries x_k chosen in the set of the q roots of unity, with uniform probability. All moments $\langle x_k^r\rangle$ have values 1 or 0, according to r being a multiple of q or not. Therefore, only walks that visit each site a number of multiples of q do contribute, and with weight one, to tr $M(\mathbf{1},\boldsymbol{x})^p$. This nonlocal constraint strongly modifies the statistical properties of the random walks, as was shown by Noh *et al* [9] in a model of surface growth. The case $q = 2$ of 'even-visiting walks' has been investigated in great detail by Bauer *et al* [10], while the limit case $q = \infty$ was considered by Derrida *et al* [8].

On the other hand, given an ensemble of bidiagonal random matrices $M(\boldsymbol{a},\boldsymbol{b})$, the enumeration methods may be useful for the study of the spectral density, usually in the limit $N \to \infty$. If all random numbers $x_j = a_j b_j$ are positive, the matrices are similar to real symmetric ones, $M(\boldsymbol{c},\boldsymbol{c})$, with $c_j = \sqrt{x_j}$. The spectral density has support on the real axis and can be achieved by the usual resolvent approach. The average resolvent can be obtained by evaluating the ensemble average tr$M(\mathbf{1},\boldsymbol{x})^p$ for every p. To this end, the random walk description could prove to be a valuable tool, even for approximate results, if one were able to acknowledge and count the dominant walks. In the case where the random variables x_j are unrestricted, as in [11], the spectral density has complex support, and the evaluation of the resolvent may only allow the determination of the boundary of the support.

2. Counting by iterations along the length of the path

One is often interested in the enumeration of paths according to a parameter having the additive property: the parameter of a sequence of Dyck paths is the sum, or a linear combination, of the parameters of the single Dyck paths in the sequence (see, for instance, [12]). Perhaps the best known example of an additive parameter is the number of returns to the origin, see [26]. The method of enumeration by iteration along the length is suitable for the case of additive parameters, and it is described here following the pattern of [26].

2.1. Statistics of peaks and valleys

Given a walk we say that an 'inversion' occurs at time k if $\mu_k \neq \mu_{k+1}$. One may further partition inversions into peaks ($\mu_k = 1$, $\mu_{k+1} = -1$) and valleys ($\mu_k = -1$, $\mu_{k+1} = 1$). We now show that the evaluation of the number of inversions in simple random walks is a straightforward combinatorial exercise rather similar to the evaluation of the number of visits to the origin, which is known in the literature. The counting of inversions in long walks will be expressed in terms of the counting in shorter and simpler ones.

It is useful to code a simple random walk of n steps as a sequence of m positive integers $n_1 \ldots n_m$ that count consecutive equal steps. A walk with initial step $\mu_1 = 1$ and described by

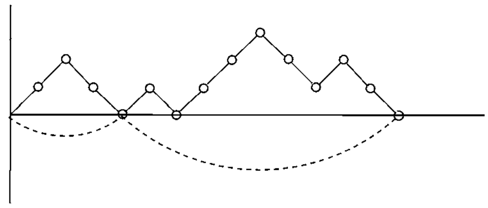


Figure 2. A weakly positive closed walk is either strictly positive or it is a strictly positive closed walk followed by a shorter weakly positive one.

the sequence $(n_1, \ldots, n_m)$ begins with n_1 steps $\mu = +1$, followed by n_2 steps in the opposite direction $\mu = -1$, and so on. The sum $n_1 + \cdots + n_m$ is the length n of the walk, and the number of inversions is $m - 1$. For instance, the walk of 12 steps and 3 inversions in figure 1 is coded by the sequence $(3, 4, 2, 3)$. The same sequence also describes a second walk, with opposite signs of steps.

Each sequence $(n_1, n_2, \ldots, n_m)$ of positive integers is a composition of the integer n in m parts and the number of such compositions is $\mathcal{C}(n, m)$:

$$\mathcal{C}(n, m) = \begin{pmatrix} n-1 \\ m-1 \end{pmatrix}. \tag{2.1}$$

It follows that the number of random paths of n steps and m inversions is $2\,\mathcal{C}(n, m+1)$. By summing over inversions one reproduces the total number of random paths with n steps, $2\sum_m \mathcal{C}(n, m) = 2^n$.

Our first evaluation is the number of walks of given length and given number of inversions. It is convenient to consider first walks returning to the origin (therefore the number of steps n is even) and introduce the counting numbers:

- $c_1(n, m)$ is the number of strictly positive closed walks of n steps and m inversions, $m = 1, 3, \ldots, n-3$
- $c_2(n, m)$ is the number of weakly positive closed walks of n steps and m inversions, $m = 1, 3, \ldots, n-1$
- $c_3(n, m)$ is the number of closed walks of n steps and m inversions, with no positivity requirement, $m = 1, 2, \ldots, n-1$.

Since for strictly positive walks it is $\mu_1 = \mu_2 = 1$ and $\mu_{n-1} = \mu_n = -1$, and for weakly positive walks it is $\mu_1 = 1$ and $\mu_n = -1$, it follows that

$$\begin{aligned} c_1(n, m) &= c_2(n-2, m) \qquad \text{for} \quad n \geqslant 2 \\ c_1(2, m) &= c_2(2, m) = \delta_{m,1}. \end{aligned} \tag{2.2}$$

A walk contributing to $c_2(n, m)$ is either in the class contributing to $c_1(n, m)$ or has at least one visit to the origin before the last site (figure 2). If a walk contributing to c_2 but not to c_1 first returns to the origin after n_1 steps, we have

$$c_2(n, m) = c_1(n, m) + \sum_{\substack{n_1=2,4,\ldots,n-2 \\ m_1=3,5,\ldots,n_1-3}} c_1(n_1, m_1)\, c_2(n - n_1, m - m_1 - 1). \tag{2.3}$$

In terms of generating functions

$$C_i(x, y) = \sum_{\substack{n=2,4,\ldots \\ m=1,2,\ldots,n-1}} c_i(n, m)\, x^n y^m \tag{2.4}$$

equations (2.2) and (2.3) translate into

$$\begin{aligned} C_1(x, y) &= x^2 y + x^2 C_2(x, y) \\ C_2(x, y) &= C_1(x, y) + y C_1(x, y) C_2(x, y). \end{aligned} \tag{2.5}$$

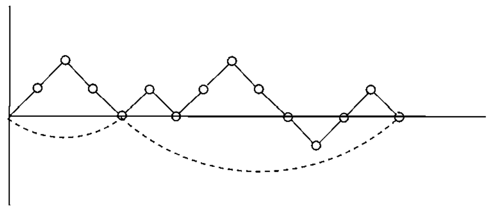


Figure 3. A closed walk is a strictly positive (or strictly negative) closed walk followed by a shorter closed walk.

The resulting algebraic quadratic equations lead to

$$C_1(x, y) = \frac{1 - x^2 + x^2y^2 - \sqrt{(1 - x^2 + x^2y^2)^2 - 4x^2y^2}}{2y} \tag{2.6}$$

$$C_2(x, y) = \frac{1 - x^2 - x^2y^2 - \sqrt{(1 - x^2 + x^2y^2)^2 - 4x^2y^2}}{2x^2y}. \tag{2.7}$$

The generating functions are closely related to the function $u(x, y)$ evaluated by Narayana [27]

$$\begin{aligned} u(x, y) &= 1 - \sqrt{(1 - x + xy)^2 - 4xy} \\ &= x + xy + \sum_{n=2}^{\infty}\sum_{r=1}^{n-1} x^n y^r \frac{2}{n-1}\binom{n-1}{r}\binom{n-1}{n-r}. \end{aligned} \tag{2.8}$$

For example, it is simple to obtain

$$C_2(x, y) = \sum_{n=2}^{\infty}\sum_{m=1}^{n-1} x^{2n-2}y^{2m-1}\frac{1}{n-1}\binom{n-1}{m}\binom{n-1}{n-m}. \tag{2.9}$$

A walk contributing to $c_3(n, m)$ (an example is given in figure 3) is either strictly positive or negative, or it has a first return to the origin after n_1 steps, where it may or may not have an inversion.

This leads to the equation

$$C_3(x, y) = 2C_1(x, y) + (1 + y)C_1(x, y)C_3(x, y). \tag{2.10}$$

The solution is found using the previous results and is written in the convenient form

$$C_3(x, y) = -1 + \frac{1 - x^2(1 - y)^2}{\sqrt{(1 - x^2 + x^2y^2)^2 - 4x^4y^2}}. \tag{2.11}$$

To extract the coefficients of the power expansion we use the generating function of Legendre polynomials, also found in [27]:

$$\begin{aligned} \frac{1}{\sqrt{(1 - x^2 - x^2y^2)^2 - 4x^4y^2}} &= \sum_{n\geqslant 0} x^{2n}(y^2 - 1)^n P_n\left(\frac{y^2+1}{y^2-1}\right) \\ &= \sum_{n,m}\binom{n}{m}^2 x^{2n}y^{2m}. \end{aligned} \tag{2.12}$$

Then

$$C_3(x, y) = 2x^2y + 2\sum_{n\geqslant 2, m\geqslant 1} x^{2n}\left[y^{2m-1}\binom{n-1}{m-1}^2 + y^{2m}\binom{n-1}{m}\binom{n-1}{m-1}\right]. \tag{2.13}$$

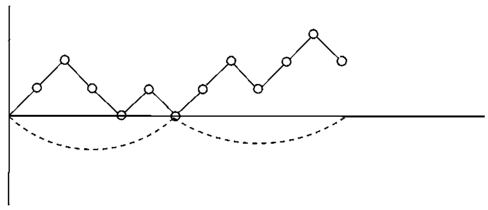


Figure 4. A weakly positive walk is a weakly positive closed walk followed by a strictly positive one.

Equation (2.13) has a simple interpretation when the paths are coded by the sequences mentioned at the beginning of this section. A random path with n steps and m inversions contributing to $c_3(n, m)$ is coded by a sequence $(n_1, n_2, \ldots, n_{m+1})$ such that $\sum_{j \text{ odd}} n_j = \sum_{j \text{ even}} n_j = n/2$. The two subsequences $s_{\text{odd}} = (n_1, n_3, \ldots)$ and $s_{\text{even}} = (n_2, n_4, \ldots)$ both have $(m + 1)/2$ terms if m is odd, whereas if m is even s_{odd} has $\frac{m}{2} + 1$ terms and s_{even} has $m/2$ terms. Then, if m is odd, the number of sequences $(n_1, n_2, \ldots, n_{m+1})$ is given by $\mathcal{C}(\frac{n}{2}, \frac{m}{2} + 1)^2$, whereas if m is even it is given by the product $\mathcal{C}(\frac{n}{2}, \frac{m}{2} + 1)\ \mathcal{C}(\frac{n}{2}, \frac{m}{2})$. Since each sequence describes two walks, one obtains the coefficients $c_3(n, m)$ in equation (2.13). Analogous results are quoted in the monograph [28].

Let us now consider random walks of n steps and m inversions without the restriction that the end point of the path is the origin, and no restriction about positivity. Such walks correspond to all sequences $(n_1, n_2, \ldots, n_{m+1})$ with $n_1 + \cdots + n_{m+1} = n$. From equation (2.1) we obtain the generating function of the counting numbers:

$$C_4(x, y) = 2 \sum_{n,m} x^n\, y^m \binom{n-1}{m} = \frac{2}{(1 + y)\,[1 - x(1 + y)]}. \tag{2.14}$$

Finally we evaluate the number $c_5(n, m)$ of weakly positive walks of n steps and m inversions, which may not be closed; an example is given in figure 4. The corresponding generating function is $C_5(x, y)$.

We may determine $c_5(n, m)$ in terms of shorter walks, by partitioning the set of weakly positive walks into three disjoint sets: (a) the set of walks that, after an initial upward step, never return to level one, (b) the set of walks with an initial upward step followed by a walk in the ensemble described by the generating function $C_2(x, y)$; (c) the set of walks of type (b) followed by a generic walk of type $C_5(x, y)$. This translates into the equation

$$C_5(x, y) = x + x\, C_5(x, y) + x\, C_2(x, y) + xy\, C_2(x, y)\, C_5(x, y). \tag{2.15}$$

Then

$$\begin{aligned} C_5(x, y) &= -\frac{1 + x - xy}{2} + \frac{1}{2} \frac{\sqrt{(1 - x^2 + x^2y^2)^2 - 4x^2y^2}}{1 - x - xy} \\ &= xy + \frac{1 + x + xy}{2}\, C_3(x, y). \end{aligned} \tag{2.16}$$

The last equality allows an easy derivation of the counting numbers $c_5(n, m)$. As a simple check, we notice that the function $C_5(x, y = 1)$ enumerates walks of fixed length and any number of inversions:

$$C_5(x, y = 1) = \frac{1}{2}\left(- 1 + \sqrt{\frac{1 + 2x}{1 - 2x}} \right). \tag{2.17}$$

The numbers are well known, see for instance [26].

2.2. Statistics of the area

The approach illustrated in the previous paragraph can also be used to obtain equations for the generating functions that enumerate the positive random walks of T steps returning to the origin and enclosing with the time axis a fixed area A. This problem was recently discussed by Jonsson and Wheater, and we refer to their paper [23] for the physics motivations of the study and the analysis of the functional equation for the generating function.

Let $c_1(A, T)$ and $c_2(A, T)$ be the numbers of, respectively, strictly and weakly positive walks that return to the origin after T steps, such that the area between the walk and the horizontal axis is A. In both cases T is an even integer. Just as in figure 2 and equation (2.3) we have

$$c_2(A, T) = c_1(A, T) + \sum_{\substack{A_1=1,2,\ldots \\ T_1=2,4,\ldots}} c_1(A_1, T_1)\, c_2(A - A_1, T - T_1). \tag{2.18}$$

Any weakly positive path may be elevated by adding a step at the beginning and another at the end, thus transforming it into a strictly positive path. Then

$$\begin{aligned} &c_2(A, T) = c_1(A + T + 1, T + 2) \qquad \text{for} \quad A \geqslant 2 \\ &c_2(A = 1, T) = c_1(A = 1, T) = \delta_{T,2}. \end{aligned} \tag{2.19}$$

We define the generating functions

$$F_i(x, y) = \sum_{A,T} c_i(A, T)\, x^A\, y^T \qquad \text{for} \quad i = 1, 2. \tag{2.20}$$

Then equations (2.18) and (2.19) translate into

$$\begin{aligned} &F_2(x, y) = F_1(x, y) + F_1(x, y)\, F_2(x, y) \\ &F_1(x, y) = x\, y^2 + x\, y^2\, F_2(x, xy). \end{aligned} \tag{2.21}$$

One obtains the nonlinear functional equation of Jonsson and Wheater:

$$F_1(x, y) = x\, y^2 + F_1(x, y)\, F_1(x, xy) \tag{2.22}$$

which has a formal solution as an infinite continued fraction:

$$F_1(x, y) = \cfrac{xy^2}{1 - \cfrac{x^3y^2}{1-\cfrac{x^5y^2}{1-\cdots}}}. \tag{2.23}$$

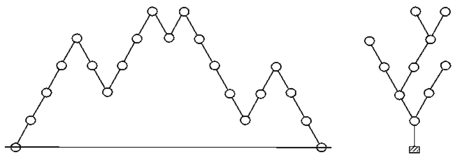
In a similar way we obtain the functional equation for $F_2(x, y)$:

$$1 - xy^2[1 + F_2(x, y)] = \frac{1}{1 + F_2(x, xy)} \tag{2.24}$$

$$F_2(x, y) = -1 + 1\cfrac{1}{1 - \cfrac{x\,y^2}{1-\cfrac{x^3\,y^2}{1-\cdots}}}. \tag{2.25}$$

From equations (2.22) and (2.24), one has $F_1(1, y) = \frac{1-\sqrt{1-4y^2}}{2}$ and $F_2(1, y) = \frac{1-2y^2-\sqrt{1-4y^2}}{2y^2}$. As shown in [23], equation (2.22) allows the evaluation of $\frac{\partial}{\partial x} F_1(x, y)|_{x=1}$, which is the generating function for the area associated with the set of strictly positive paths of length T:

$$\frac{\partial}{\partial x} F_1(x, y)|_{x=1} = \sum_T y^T \sum_A A\, c_1(A, T) = \frac{y^2}{1 - 4y^2}. \tag{2.26}$$



j	N_j	n_j
5	2	2
4	4	2
3	5	3
2	5	2
1	3	1
0	2	0

Figure 5. The one-to-one map between strictly positive closed walks and rooted planar trees.

Table 1. Corresponding elements in the one-to-one map between strictly positive closed paths and rooted planar trees, indicated in figure 5.

Dyck path	Planar rooted tree
One pair of matched up–down steps	One edge of the tree
length of path $= 2n$	Tree has n edges
N_j, $j = 1, \ldots, t$	Sum of degrees of vertices at level j
n_j, $j = 1, \ldots, t$	Number of edges between level $j-1$ and level j
$N_j - n_{j+1}$, $j = 1, \ldots, t$	$N_j - n_{j+1} = v_j =$ number of vertices at level j
Height t	Height t

In the same way we obtain the generating function for the area associated with the set of weakly positive paths of length T:

$$\frac{\partial}{\partial x} F_2(x, y)|_{x=1} = \sum_T y^T \sum_A A\, c_2(A, T) = \frac{1 - 2y^2 - \sqrt{1 - 4y^2}}{2y^2(1 - 4y^2)}. \qquad (2.27)$$

The results (2.26) and (2.27) were obtained in [24] and extended to higher moments of the area in [25] by a different method. We do not proceed to the analysis of equation (2.22), whose main properties were found in [23]. In the next section we describe a different approach to enumerate random walks, which will provide a different viewpoint of this problem.

3. Counting by insertions which increase the width of the path

The counting method described in section 2 was based on the construction of the ensemble of random walks by adding new random walks at the end of previously considered ones. We now present a different approach, where the ensemble is generated by inserting new random walks on the vertices of previously considered ones.

To evaluate the number of Dyck paths of $2n$ steps, height t, that visit the sites $j = 0, 1 \ldots t$ prescribed numbers N_j of times, it is useful to consider the one-to-one map between Dyck paths and planar rooted trees [20]. The correspondence is shown in figure 5 where the rooted planar tree on the right side is mapped to the compressed 'mountain range' on the left side, where each edge of the tree arises from one matched up–down pair.

Given a Dyck path, let n_j denote the number of up steps between level $j-1$ and level j. Since N_j is the sum of the number of up steps from level $j-1$ and level j plus the number of down steps between level j and level $j+1$, we have

$$N_j = n_j + n_{j+1} \qquad j = 1, \ldots, t \qquad n_0 = 0 \qquad N_0 = n_1 + 1. \qquad (3.1)$$

Table 1 lists the corresponding elements in the one-to-one map between strictly positive closed paths and rooted planar trees.

The number $N(v_1, v_2, \dots, v_t)$ of planar rooted trees with a given set of non-root vertices $\{v_1, \dots, v_t\}$ is evaluated in a recursive way because planar rooted trees with a set of non-root vertices $\{v_1, \dots, v_t, v_{t+1}\}$ are obtained by adding v_{t+1} new vertices at level $t+1$ and using $t+1$ new edges to connect all of them to some of the v_t vertices at level t. This may be done in $\binom{v_{t+1}+v_t-1}{v_{t+1}}$ ways. Then

$$N(v_1, v_2, \dots, v_t) = \binom{v_2 + v_1 - 1}{v_2}\binom{v_3 + v_2 - 1}{v_3}\dots\binom{v_t + v_{t-1} - 1}{v_t}. \tag{3.2}$$

Since $v_j = N_j - n_{j+1} = n_j$ the evaluation (3.2) provides the number $N(n_1, n_2, \dots, n_t)$ of Dyck paths of height t, with specified numbers n_j of up steps between level $j-1$ and level j:

$$N(n_1, n_2, \dots, n_t) = \binom{n_1 + n_2 - 1}{n_2}\binom{n_2 + n_3 - 1}{n_3}\dots\binom{n_{t-1} + n_t - 1}{n_t}. \tag{3.3}$$

Such paths have total length $2n$, where $n = n_1 + \dots + n_t$.

We remark that it might seem more natural to enumerate Dyck paths of fixed length in terms of the integers N_j enumerating the visits at each level, rather than in terms of the positive integers n_j enumerating the up steps from each level. The second coding is superior because the positive numbers n_j are independent variables, save for the restriction $\sum n_j = n$.

We obtained the result (3.3) by inserting new mountain ranges on top of the highest vertices of Dyck paths [11]. We were unaware of the previous works on the enumeration of vertices in planar rooted trees [13, 21].

We now proceed to discuss some properties of the enumeration numbers (3.3) by introducing generating functions. They have a very direct relation with general tridiagonal matrices. This fruitful and interesting connection was described long ago by Flajolet [32] in his basic paper, apparently almost ignored in later investigations.

Let $F_t(x_1, x_2, \dots, x_t)$ be the generating function for the numbers $N(n_1, \dots, n_t)$:

$$F_t(x_1, x_2, \dots, x_t) = \sum_{n_j=1,\dots,\infty,\, j=1,\dots,t} N(n_1, n_2, \dots, n_t)\, x_1^{n_1} x_2^{n_2} \dots x_t^{n_t}. \tag{3.4}$$

We derive various useful properties of the function here that originate from a recursive relation.

Proposition 1.

$$F_t(x_1, x_2, \dots, x_t) = F_{t-1}\left(x_1, x_2, \dots, \frac{x_{t-1}}{1 - x_t}\right) - F_{t-1}(x_1, x_2, \dots, x_{t-1}). \tag{3.5}$$

Proof.

$$\begin{aligned}
&\sum_{n_j=1,\dots,\infty,\, j=1,\dots,t} N(n_1, n_2, \dots, n_t) x_1^{n_1} \dots x_n^{n_t} \\
&\quad = \sum_{n_j=1,\dots,\infty,\, j=1,\dots,t-1} N(n_1, n_2, \dots, n_{t-1}) x_1^{n_1} x_2^{n_2} \dots x_{t-1}^{n_{t-1}} \\
&\qquad \times \sum_{n_t=1}^{\infty} \binom{n_t + n_{t-1} - 1}{n_t} x_t^{n_t} \\
&\quad = \sum_{n_j=1,\dots,\infty,\, j=1,\dots,t-1} N(n_1, n_2, \dots, n_{t-1}) x_1^{n_1} x_2^{n_2} \dots x_{t-1}^{n_{t-1}} \left[(1 - x_t)^{-n_{t-1}} - 1\right] \\
&\quad = F_{t-1}\left(x_1, x_2, \dots, \frac{x_{t-1}}{1 - x_t}\right) - F_{t-1}(x_1, x_2, \dots, x_{t-1}).
\end{aligned}$$

□

Beginning with $N(n) = 1$, one iteratively constructs the first few generating functions:

$$F_1(x) = \frac{x}{1-x} \qquad F_2(x,y) = \frac{xy}{(1-x)(1-x-y)}$$
$$F_3(x,y,z) = \frac{xyz}{(1-x-y)(1-x-y-z+xz)} \qquad \ldots .$$

They suggest the following formal solution to equation (3.5):

Proposition 2.

$$\begin{aligned} F_t(x_1, x_2, \ldots, x_{t-1}, x_t) &= \frac{x_1 x_2 \ldots x_{t-1} x_t}{P_t(x_1,\ldots,x_t) P_{t-1}(x_1,\ldots,x_{t-1})} \\ &= \frac{P_{t-1}(x_2,\ldots,x_t)}{P_t(x_1,\ldots,x_t)} - \frac{P_{t-2}(x_2,\ldots,x_{t-1})}{P_{t-1}(x_1,\ldots,x_{t-1})} \end{aligned} \tag{3.6}$$

where $P_t(x_1,\ldots,x_t)$ is the polynomial generated through the recursion

$$\begin{aligned} &P_t(x_1,\ldots,x_t) = P_{t-1}(x_1,\ldots,x_{t-1}) - x_t P_{t-2}(x_1,\ldots,x_{t-2}) \\ &P_0 = 1 \qquad P_1(x) = 1 - x. \end{aligned} \tag{3.7}$$

Proof. The first equality is proven by direct substitution of the solution into the recursive formula (3.5) and by using the following identity, which is proven by repeated use of equation (3.7):

$$P_t(x_1,\ldots,x_t) = (1-x_t) P_{t-1}\left(x_1,\ldots,x_{t-2}, \frac{x_{t-1}}{1-x_t}\right). \tag{3.8}$$

To prove the second equality in equation (3.6), we formally solve the recurrence relation for polynomials, with the given initial conditions, by means of a transfer matrix:

$$\begin{aligned} &\begin{pmatrix} P_t(x_1,\ldots,x_t) & P_{t-1}(x_2,\ldots,x_t) \\ P_{t-1}(x_1,\ldots,x_{t-1}) & P_{t-2}(x_2,\ldots,x_{t-1}) \end{pmatrix} \\ &\quad = \begin{pmatrix} 1 & -x_t \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -x_{t-1} \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} 1 & -x_1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{3.9}$$

By taking the determinant of both sides, we obtain a useful identity:

$$P_t(x_1,\ldots,x_t) P_{t-2}(x_2,\ldots,x_{t-1}) - P_{t-1}(x_1,\ldots,x_{t-1}) P_{t-1}(x_2,\ldots,x_t) = -x_1 x_2 \ldots x_t. \tag{3.10}$$

The equality follows by dividing both terms by $P_t(x_1,\ldots,x_t) P_{t-1}(x_1,\ldots,x_{t-1})$. □

Note that the recursive property implies that the polynomial $P_t(x_1,\ldots,x_t)$ coincides with the determinant of a matrix of size $t+1$:

$$P_t(x_1,\ldots,x_t) = \det[I + M(\mathbf{1}, \boldsymbol{x})]. \tag{3.11}$$

The generating function for walks with height not greater than t is

$$\begin{aligned} \Phi_t(x_1,\ldots,x_t) &= 1 + F_1(x_1) + F_2(x_1,x_2) + \cdots + F_t(x_1,\ldots,x_t) \\ &= \frac{P_{t-1}(x_2,\ldots,x_t)}{P_t(x_1,\ldots,x_t)}. \end{aligned} \tag{3.12}$$

This function has a simple representation in terms of the bidiagonal matrix $M(\mathbf{1}, \boldsymbol{x})$, of size $t+1$:

$$\Phi_t(x_1, x_2, \ldots, x_t) = [I + M(\mathbf{1}, \boldsymbol{x})]^{-1}_{1,1}. \tag{3.13}$$

By replacing the last term in the sum (3.12) with the right side of equation (3.5) and resumming, one obtains the recursive property

$$\Phi_t(x_1, x_2, \ldots, x_{t-1}, x_t) = \Phi_{t-1}\left(x_1, x_2, \ldots, \frac{x_{t-1}}{1-x_t}\right) \tag{3.14}$$

and also, since $\Phi_1(x) = (1-x)^{-1}$, the following finite continued fraction representation:

$$\Phi_t(x_1, x_2, \ldots, x_t) = \frac{1}{1-\frac{x_1}{1-\frac{x_2}{1-\ldots\frac{x_{t-1}}{1-x_t}}}}. \tag{3.15}$$

The numbers $N(n_1, \ldots, n_t)$ provide very detailed information about random walks. One is often interested in reduced information, which requires restricted summations over the variables $n_1 \ldots n_t$. This is, in general, a rather formidable task. We here provide some interesting examples. In the first we count closed weakly positive walks of fixed height and length and obtain identities involving sums of products of binomials. In the second example, we again discuss the problem of the area, by relating it to certain tridiagonal matrices. The third example addresses the issue of sums of weighted walks.

Example 1 (Enumeration of walks with fixed length and height). Let $C(n, t)$ be the number of weakly positive closed walks of length $2n$ and height t. Evidently, the parameter t cannot be larger than n. The sum over possible t values gives the well known number of weakly positive walks of length $2n$:

$$\sum_{t=1}^{n} C(n, t) = \frac{1}{n+1}\binom{2n}{n}. \tag{3.16}$$

The generating function for the numbers $C(n, t)$ can be obtained by setting all arguments x_i equal to x in the function (3.4):

$$\begin{aligned} F_t(x) &= \sum_{n=t,\ldots,\infty} x^n C(n,t) = \frac{x^t}{P_t(x)\, P_{t-1}(x)} = \frac{P_{t-1}(x)}{P_t(x)} - \frac{P_{t-2}(x)}{P_{t-1}(x)} \\ C(n,t) &= \sum_{\kappa(n;t)} N(n_1 \ldots n_t) \qquad (n \geqslant t) \end{aligned} \tag{3.17}$$

where, for brevity, we adopt the notation for the sum over the t integers $n_i > 1$, such that $n_1 + \cdots + n_t = n$ as a sum over the set $\kappa(n; t)$ of compositions of n into t integers $n_i \geqslant 1$. The polynomials $P_t(x)$ are easily evaluated from the recursion property

$$P_t(x) = P_{t-1}(x) - x P_{t-2}(x) \qquad P_0 = 1 \qquad P_1(x) = 1 - x \tag{3.18}$$

and have a simple expression in terms of the roots of the equation $z^2 - z + x = 0$:

$$\begin{aligned} P_t(x) &= \frac{z_1^{t+2} - z_2^{t+2}}{z_1 - z_2} = \frac{1}{\sqrt{1-4x}}\left[\left(\frac{1+\sqrt{1-4x}}{2}\right)^{t+2} - \left(\frac{1-\sqrt{1-4x}}{2}\right)^{t+2}\right] \\ &= \prod_{k=1}^{[(t+1)/2]}\left(1 - 4x\cos^2\frac{k\pi}{t+2}\right). \end{aligned} \tag{3.19}$$

For the last equality we used the known eigenvalues of the matrix $M(\mathbf{1}, \mathbf{1})$. $P_t(x)$ is related to a Chebyshev polynomial of the second kind.

The multiple sums over the compositions in equation (3.17) are obtained by extracting the coefficient of x^n from the generating function $F_t(x)$. This is simple for small t; for example

$$C(n,2) = \sum_{\kappa(n;2)} \binom{n_2+n_1-1}{n_2} = 2^{n-1} - 1$$
$$C(n,3) = \sum_{\kappa(n;3)} \binom{n_2+n_1-1}{n_2}\binom{n_3+n_2-1}{n_3} = -2^{n-1} + \left(\frac{3+\sqrt{5}}{2}\right)^{n-2}\left(1+\frac{2}{\sqrt{5}}\right) + \left(\frac{3-\sqrt{5}}{2}\right)^{n-2}\left(1-\frac{2}{\sqrt{5}}\right). \tag{3.20}$$

We now consider the general case.

Proposition 3.

$$C(n,t) = -\frac{1}{t+2}\sum_j \frac{1-4a_j}{a_j^{n+1}} + \frac{1}{t+1}\sum_j \frac{1-4b_j}{b_j^{n+1}} \tag{3.21}$$

where $n \geqslant t$, $\{a_j\}$ *and* $\{b_j\}$ *are, respectively, the zeros of* $P_t(x)$ *and* $P_{t-1}(x)$.

Proof. We use the following identity for the inverse of a polynomial $P(x)$ with simple roots x_j:

$$\frac{1}{P(x)} = \sum_{j=1}^{n} \frac{1}{P'(x_j)}\frac{1}{x-x_j}$$

to obtain an expression for the inverse of the polynomial $P_t(x)P_{t-1}(x)$ in terms of the known roots of the factors

$$\frac{1}{P_t(x)P_{t-1}(x)} = \sum_j \frac{1}{P_t'(a_j)P_{t-1}(a_j)}\frac{1}{x-a_j} + \sum_j \frac{1}{P_t(b_j)P_{t-1}'(b_j)}\frac{1}{x-b_j}.$$

We then use the following results, which will be proven:

$$P_t'(a_j)P_{t-1}(a_j) = (t+2)\frac{a_j^t}{1-4a_j} \qquad P_t(b_j)P_{t-1}'(b_j) = -(t+1)\frac{b_j^t}{1-4b_j} \tag{3.22}$$

and obtain the power expansion of $F_t(x)$ we searched for:

$$\frac{x^t}{P_t(x)P_{t-1}(x)} = \frac{1}{t+2}\sum_j \frac{1-4a_j}{a_j^t}\frac{x^t}{x-a_j} - \frac{1}{t+1}\sum_j \frac{1-4b_j}{b_j^t}\frac{x^t}{x-b_j}$$
$$= \sum_{n=t}^{\infty} x^n \left[-\frac{1}{t+2}\sum_j \frac{1-4a_j}{a_j^{n+1}} + \frac{1}{t+1}\sum_j \frac{1-4b_j}{b_j^{n+1}}\right].$$

To derive the first identity in equation (3.22) one evaluates $P'(x)$ in terms of the roots z_1 and z_2 of the equation $z^2 - z + x = 0$. For $x = a_j$ it is

$$P_t'(a_j) = -(t+2)\frac{z_1^{t+1}+z_2^{t+1}}{(z_1-z_2)^2}.$$

The condition $P_t(a_j) = 0$ implies that $z_1^{t+2} = z_2^{t+2}$; then

$$P_t'(a_j)P_{t-1}(a_j) = -(t+2)\frac{z_1^{2t+2}-z_2^{2t+2}}{[z_1-z_2]^3} = -(t+2)\frac{[z_1\cdot z_2]^t\,[z_2^2-z_1^2]}{[z_1-z_2]^3}.$$

This, together with $z_1+z_2 = 1$ and $z_1z_2 = a_j$, allows the simplification that yields the result. The second identity is proven along the same lines. □

The structure of the coefficients $C(n,t)$ as a two-term difference suggests the introduction of the numbers $C(n, \leqslant t)$ of walks of length $2n$ and height not larger than t:

$$\begin{aligned} C(n,\leqslant t) &= C(n,1)+C(n,2)+\cdots+C(n,t) = -\frac{1}{t+2}\sum_j \frac{1-4a_j}{a_j^{n+1}} \\ &= \frac{2}{t+2}\sum_{j=1}^{t+1}\left(4\cos^2\frac{j\pi}{t+2}\right)^n \sin^2\frac{j\pi}{t+2} \qquad (3.23)\\ &= 2\sum_{\ell=-[\frac{n}{t+2}]}^{[\frac{n}{t+2}]}\binom{2n}{n+\ell(t+2)} - \frac{1}{2}\sum_{\ell=-[\frac{n+1}{t+2}]}^{[\frac{n+1}{t+2}]}\binom{2n+2}{n+1+\ell(t+2)} \qquad (3.24)\end{aligned}$$

where we inserted the explicit form for a_j from equation (3.19). The form of equation (3.23) may be recognized from the property $C(n,\leqslant t) = [M(\mathbf{1},\mathbf{1})^{2n}]_{11}$, where $M(\mathbf{1},\mathbf{1})$ has size $t+1$. Of course $C(n,t) = C(n,\leqslant t) - C(n,\leqslant t-1)$. Note that $C(n,\leqslant n)$ is precisely the number in equation (3.16).

The generating function $\Phi_t(x)$ for the numbers is obtained by setting all arguments equal to x in the function (3.13):

$$\Phi_t(x) = \sum_{n=0}^{\infty} C(n,\leqslant t)\, x^n = \frac{2}{t+2}\sum_{j=1}^{t+1}\frac{\sin^2\frac{j\pi}{t+2}}{1-4x\cos^2\frac{j\pi}{t+2}}. \qquad (3.25)$$

Example 2 (Enumeration of walks of fixed length and area). The area of closed weakly positive walks with numbers of upward steps $n_1,\ldots,n_t$ is $A = n_1+3n_2+5n_3+\cdots+(2t-1)n_t$, while the total length is $T = 2n$, $n = n_1+\cdots+n_t$. With the position $x_1 = xy$, $x_2 = xy^3, \ldots, x_t = xy^{2t-1}$ in equation (3.4) we obtain the generating function $F_t(x,y)$ of counting numbers $C(n,t,A)$ of positive closed walks with fixed length $2n$, height t and area A:

$$\begin{aligned} F_t(x,y) &= \sum_{n,A} x^n y^A\, C(n,t,A) \\ &= \frac{P_{t-1}(xy^3,\ldots,xy^{2t-1})}{P_t(xy,\ldots,xy^{2t-1})} - \frac{P_{t-2}(xy^3,\ldots,xy^{2t-3})}{P_{t-1}(xy,\ldots,xy^{2t-3})} \qquad (3.26)\\ C(n,t,A) &= \sum_{\kappa(n;t)} N(n_1,n_2,\ldots,n_t)\,\delta\left(\sum_k (2k-1)n_k - A\right)\end{aligned}$$

where the δ function imposes the restriction that the integers $n_1 \ldots n_t$ of the compositions of n should correspond to walks with fixed area A.

If we relax the restriction on height, being interested in counting numbers $C(n,A)$ for the length and the area, the generating function is

$$\begin{aligned}\Phi_\infty(x,y) &= 1+\sum_{n,A} C(n,A)x^n y^A = 1+\sum_{t\geqslant 1} F_t(x,y) \\ &= \frac{P_\infty(xy^3,xy^5,\ldots)}{P_\infty(xy,xy^3,\ldots)}. \qquad (3.27)\end{aligned}$$

We shall solve the counting problem by giving first the explicit expression of the polynomial $P_t(x,y) = P_t(xy,xy^3,\ldots,xy^{2t-1})$, which coincides with the determinant of the tridiagonal matrix of size $t + 1$:

$$I + M(\mathbf{1}, x\boldsymbol{y}) = \begin{pmatrix} 1 & xy & & & & \\ 1 & 1 & xy^3 & & & \\ & 1 & 1 & xy^5 & & \\ & & \ldots & \ldots & \ldots & \\ & & & 1 & 1 & xy^{2t-1} \\ & & & & 1 & 1 \end{pmatrix}. \tag{3.28}$$

Proposition 4.

$$P_t(x, y) = 1 + \sum_{n=1}^{[(t+1)/2]} (-1)^n x^n y^{n(2n-1)} \prod_{k=1}^{n} \frac{1 - y^{2(t-n+2-k)}}{1 - y^{2k}}. \tag{3.29}$$

Proof. The polynomials solve the recurrence relation $P_t(x, y) = P_{t-1}(x, y) - xy^{2t-1} P_{t-2}(x, y)$ with $P_0 = 1$ and $P_1(x, y) = 1 - xy$. The explicit computation of the first few ones suggests the following general structure: $P_t(x, y) = \sum_{n=0}^{[(t+1)/2]} (-x)^n c_{n,t}(y)$, where $c_{n,t}(y)$ are polynomials for which recurrence relations are easily written. By solving the first few ones, one easily arrives at the conjecture in the form stated in the proposition. A formal proof then follows by induction, but we omit it. □

For the purpose of the statistics of the area, we need the limit $t \to \infty$:

$$P_\infty(x, y) = 1 + \sum_{n>0} (-x)^n \frac{y^{n(2n-1)}}{(1 - y^2)(1 - y^4) \ldots (1 - y^{2n})}. \tag{3.30}$$

From this expression, we derive the following functional equation:

$$P_\infty(x, y) - P_\infty(xy^2, y) + xy\, P_\infty(xy^4, y) = 0. \tag{3.31}$$

Proof.

$$\begin{aligned} P_\infty(xy^2, y) &= 1 + \sum_{n>0} (-x)^n \frac{y^{n(2n-1)}}{(1 - y^2) \ldots (1 - y^{2n-2})} \frac{y^{2n}}{1 - y^{2n}} \\ &= P_\infty(x, y) - \sum_{n>0} (-x)^n \frac{y^{n(2n-1)}}{(1 - y^2) \ldots (1 - y^{2(n-1)})} \\ &= P_\infty(x, y) + xy\, P_\infty(xy^4, y). \end{aligned}$$

□

Since $\Phi_\infty(x, y) = P_\infty(xy^2, y)/P_\infty(x, y)$, we obtain a functional equation which corresponds to equation (2.24) for $F_2(x, y)$:

$$\Phi_\infty(x, y) = 1 + xy\, \Phi_\infty(x, y)\, \Phi_\infty(xy^2, y). \tag{3.32}$$

The formula (3.27) allows us to compare $\Phi_\infty(x, y)$ with the generating function $F_2(x, y)$ discussed in section 2 for weakly positive walks. There, the variable y was conjugated to the total length $T = 2n$ and x was conjugated to the area. Here, for uniformity with the rest of the paper, x is conjugated to n and y to A. It turns out that $\Phi_\infty(x^2, y) = 1 + F_2(y, x)$.

Remark. From the continued fraction representation of $\Phi_\infty(x_1, x_2, x_3, \ldots)$, see equation (3.15)

$$\Phi_\infty(x_1, x_2, x_3, \ldots) = \cfrac{1}{1 - \cfrac{x_1}{1 - \cfrac{x_2}{1 - \cfrac{x_3}{1 - \cdots}}}} \tag{3.33}$$

it is easy to see that it solves the formal equation

$$\Phi_\infty(x_1, x_2, x_3, \ldots) = 1 + x_1\, \Phi_\infty(x_1, x_2, x_3, \ldots)\, \Phi_\infty(x_2, x_3, x_4, \ldots). \tag{3.34}$$

Let us suppose that each x_k is a function of two variables $x_k = x_k(x, y)$. The above formal equation will turn into a functional equation for a function $\Phi_\infty(x, y)$ if $x_k(x, y) = x_{k-1}(\overline{x}, \overline{y})$, where the replacements $x \to \overline{x}$ and $y \to \overline{y}$ do not depend on k. This is possible for the case

$$x_k = x\, y^{pk+r} \qquad p \text{ and } r \text{ real.} \tag{3.35}$$

Then equation (3.34) becomes

$$\Phi_\infty(x, y) = 1 + xy^{p+r}\, \Phi_\infty(x, y)\, \Phi_\infty(xy^p, y). \tag{3.36}$$

Example 1 in this section corresponds to $p = 0$ and $r = 0$, and example 2 corresponds to $p = 2, r = -1$. However, the more general equation (3.36) does not add substantially to the previous examples since it provides the generating function for counting Dyck paths with a given number of steps and given $f(p, r) = \sum_j (pj + r)n_j$, which is a linear combination of the area and the length of the paths: $f(p, r) = (p/2)A + (2n)(r/2 + p/4)$.

Another interesting case is

$$x_k = xy^{a^k} \tag{3.37}$$

which will be discussed in the end of the next example.

As this paper was being written we became aware of recent papers which have some overlap with this section [29–31].

Example 3 (Toy functional integrals). In the introduction we mentioned the subject of weighted Dyck paths and speculated about the possibility of an operative definition of path integral in the framework of Dyck paths. One may consider a variety of functionals $G[\gamma]$ where γ is a Dyck path of length $2n$, and write

$$\int \mathcal{D}\gamma\, G[\gamma] = \frac{1}{C(n)} \sum_{\gamma\in\Gamma(n)} G[\gamma]$$

where $\{s_k\}_{k=0}^{2n}$, with $s_0 = s_{2n} = 0$, is the sequence of sites visited by the path γ and $C(n)$ is the cardinality of the set $\Gamma(n)$ of Dyck paths with $2n$ steps. It is given by the Catalan number $C(n) = \frac{1}{n+1}\binom{2n}{n}$, see equation (3.16). Since we shall not consider the continuum limit, our finite sums are here called toy functional integrals.

We are also interested in the set $\Gamma(n, t)$ of Dyck paths of length $2n$, height t and cardinality $C(n, t)$, see proposition 3, whence one obtains the self-explanatory set $\Gamma(n, \leqslant t)$. We discuss in some detail the class of functionals $G[\gamma] = \int \mathrm{d}\tau\, g(\gamma(\tau))$ and consider the toy integral

$$\int_{\Gamma(n,t)} \mathcal{D}\gamma \int \mathrm{d}\tau\, g(\gamma(\tau)) = \frac{1}{C(n, t)} \sum_{\gamma\in\Gamma(n,t)} \sum_{k=1}^{2n} g(s_k). \tag{3.38}$$

Two interesting examples are

$$g(s) = s^r \qquad g(s) = c\, a^s \qquad a > 0. \tag{3.39}$$

In particular, the toy integral with $g(s) = s$ evaluates the average area enclosed by Dyck paths of height t and length $2n$. The second example leads to explicit equations and will be discussed after the general case.

Equation (3.38) is first written in terms of the numbers N_j of visits of site j of the Dyck path γ, and next in terms of the familiar numbers of upward steps n_j:

$$\sum_{k=1}^{2n} g(s_k) = \sum_{j=0}^{t} g(j)N_j = \sum_{j=0}^{t} g(j)(n_j + n_{j+1}) = \sum_{j=0}^{t} [g(j) + g(j-1)]n_j. \tag{3.40}$$

The sum over paths is performed by introducing the counting numbers $N(n_1, n_2, \ldots, n_t)$:

$$\sum_{\gamma\in\Gamma(n,t)} \sum_{j=1}^{t}[g(j)+g(j-1)]n_j = \sum_{\kappa(n;t)} N(n_1,n_2,\ldots,n_t)\sum_{j=1}^{t}[g(j)+g(j-1)]n_j$$

$$= \sum_{G\in\mathcal{G}} G\cdot C(n,t,G) \tag{3.41}$$

where $C(n, t, G)$ is the number of Dyck paths of length $2n$, height t and fixed value G:

$$C(n,t,G) = \sum_{\kappa(n;t)} N(n_1,n_2,\ldots,n_t)\,\delta\left(G - \sum_i [g(i)+g(i-1)]n_i\right) \tag{3.42}$$

and the sum $G \in \mathcal{G}$ runs over the set $\mathcal{G}$ of the possible values of G. The generating function

$$F_t(x,y) = \sum_{n,G} C(n,t,G)\,x^n y^G$$

is obtained from the general theory by setting $x_k = x\, y^{g(k)+g(k-1)}$ in equation (3.6). It may be evaluated from ratios of the determinants $P_t(x, y)$.

The function

$$\Phi_\infty(x,y) = 1 + F_1(x,y) + \cdots + F_t(x,y) + \cdots = \sum_{n,G} C(n,G)x^n y^G \tag{3.43}$$

is the generating function for the numbers $C(n, G) = \sum_{t=1}^{n} C(n, t, G)$ that count Dyck paths of length $2n$ and fixed value G. The generating function $f(x)$ for the numbers $G_n = \sum_{G\in\mathcal{G}} GC(n, G)$ can be written as the first derivative of $\Phi_\infty(x, y)$:

$$f(x) = \sum_{n=1}^{\infty} G_n x^n = \frac{\partial}{\partial y}\Phi_\infty(x,y)\bigg|_{y=1}. \tag{3.44}$$

While we have been unable to find a 'quadratic' functional equation for the function $\Phi_\infty(x, y)$ related to the toy integral (3.38) with $g(s) = s^r$ for $r \geqslant 2$, we were successful with the exponential function $g(s) = ca^s$. In this case the functional evaluated on a Dyck path is, see equation (3.40),

$$G = c\sum_{k=0}^{2n} a^{s_k} = c\frac{a+1}{a}\sum_{s=1}^{t} a^s n_s.$$

The generating function for counting numbers is obtained with the position $x_k = xy^{(a^k)}$, so that

$$x_1^{n_1}\ldots x_t^{n_t} = x^{n_1+\cdots+n_t} y^{n_1 a + n_2 a^2 + \cdots + n_t a^t} = x^n y^G$$

and, for the general theory, it is given by the continued fraction

$$\Phi_\infty(x,y) = \cfrac{1}{1 - \cfrac{xy^a}{1 - \cfrac{xy^{(a^2)}}{1 - \cfrac{xy^{(a^3)}}{1 - \cdots}}}}$$

which corresponds to the functional equation

$$\Phi_\infty(x,y) = 1 + xy^a \Phi_\infty(x,y)\Phi_\infty(x,y^a). \tag{3.45}$$

From this equation, since $\Phi_\infty(x, 1) = (1 - \sqrt{1-4x})/(2x)$, one easily obtains the generating function for the numbers G_n, see equation (3.44),

$$\frac{\partial}{\partial y}\Phi_\infty(x,y)\bigg|_{y=1} = \frac{a}{x}\frac{(1-2x-\sqrt{1-4x})}{(1-a+(1+a)\sqrt{1-4x})}.$$

It is interesting that the counting numbers $C(n, A)$, see equation (3.27), allow us to define a partition function for Dyck paths of any length $L = 2n$, weighted with their length L and area A:

$$\int_\Gamma \mathcal{D}\gamma\ \mathrm{e}^{-aL[\gamma]-bA[\gamma]} = \sum_{L,A} \mathrm{e}^{-aL-bA} C(L/2, A) = \Phi_\infty(\mathrm{e}^{-2a}, \mathrm{e}^{-b}) \tag{3.46}$$

where Γ is the set of all Dyck paths and a different normalization has been used.

3.1. Enumeration of closed paths

The counting numbers $N(n_1, \ldots, n_t)$ apply to weakly positive closed paths and are useful for expressing the matrix element (1, 1) of $M(\mathbf{1}, \boldsymbol{x})^{2p}$ as a polynomial in the entries x_i of the matrix, see equation (1.5).

The generalization of the counting numbers to arbitrary closed paths is straightforward, and is given in

Proposition 5. *The number of closed paths of height t, depth s and length $2p$, containing n_{-i} upward steps $(-i-1, -i)$, $i = s-1, \ldots, 0$, and n_j upward steps $(j-1, j)$, $j = 1, \ldots, t$, is*

$$\begin{aligned} &N_{s,t}(2p;\ n_{-s+1} \ldots n_0 \,|\, n_1 \ldots n_t) \\ &\quad = N(n_0, n_{-1} \ldots n_{-s+1}) \binom{n_0 + n_1}{n_1} N(n_1, n_2, \ldots, n_t) \\ &\quad = \prod_{i=1}^{s-1} \binom{n_{-i} + n_{-i+1} - 1}{n_{-i}} \binom{n_0 + n_1}{n_0} \prod_{i=1}^{t} \binom{n_i + n_{i+1} - 1}{n_{i+1}}. \end{aligned} \tag{3.47}$$

Proof. Closed paths contributing to the above counting number are sequences of n_1 closed positive paths and n_0 closed negative paths. The factors $N(n_0, \ldots, n_{-s+1})$ and $N(n_1, \ldots, n_t)$, respectively, count the negative and positive closed paths that are obtained by joining the negative or positive subsequences. The intermediate binomial factor counts the ways in which the subsequences can be ordered. □

The length $2p$ is specified in the symbol in view of a further generalization to open paths; for closed paths $p = n_{-s+1} + \cdots + n_t$.

The counting numbers allow us to write the expression for the matrix element (i, i), $i = 1 \ldots N$ of the powers of the bidiagonal matrix $M(\mathbf{1}, \boldsymbol{x})$ as a polynomial in the entries:

$$M(\mathbf{1}, \boldsymbol{x})_{ii}^{2p} = \sum_{t=0}^{N-i} \sum_{s+t=1}^{N-1} \sum_{\kappa(2p,s+t)} N_{s,t}(2p;\ n_{-s+1} \ldots n_0 \,|\, n_1 \ldots n_t) x_{i-s}^{n_{-s+1}} \ldots x_{i-1}^{n_0} x_i^{n_1} \ldots x_{i+t-1}^{n_t}. \tag{3.48}$$

3.2. Enumeration of general paths

The enumeration is extended to paths that may not return to the origin. For open paths the counting numbers $N_{s,t}(p;\ n_{-s+1}, \ldots, n_0 \,|\, n_1, \ldots, n_t)$ correspond to an expression which is more general than the above one.

The paths have length p and make the specified set of upward steps between sites $-s$ and t. Since the total number of upward steps is $u = n_{-s+1} + \cdots + n_t$ and the number of downward steps is $d = p - u$, the endpoint of the walks is site (level) $y = u - d = 2u - p$. If $2u - p \neq 0$ the paths are open.

For all paths it is $n_1, \ldots, n_t \geqslant 1$, because the walk has to reach the highest site t. However, for open paths some numbers n_i with $i < 0$ can be zero. More precisely, if $y \geqslant 0$ all $n_i \geqslant 1$

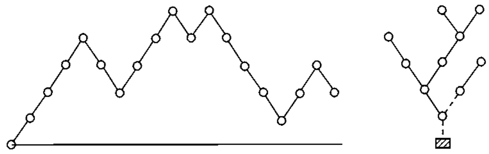


Figure 6. The one-to-one map between strictly positive walks and generalized rooted planar trees.

because the walk has to ascend from $-s$ to the level y; if $y < 0$ it may be that $n_k = 0$ for $k \leqslant y$.

With this notation and postponing the problem of the evaluation of the counting numbers, we write a general expression for the power of a matrix $M(\mathbf{1}, \boldsymbol{x})$ as a *polynomial in the entries* $x_1, \ldots, x_{N-1}$:

Proposition 6.

$$[M(\mathbf{1}, \boldsymbol{x})^p]_{i,j} = \sum_{s,t} \sum_{n_{-s+1}\ldots n_t} N_{s,t}(p;\ n_{-s+1}\ldots n_0 \,|\, n_1 \ldots n_t) x_{i-s}^{n_{-s+1}} \ldots x_{i-1}^{n_0} x_i^{n_1} \ldots x_{i+t-1}^{n_t}. \quad (3.49)$$

In the first sum s and t have the following restrictions:

(1) The finite size of the matrix, which constrains walks inside the strip of integers $1 \ldots N-1$ that label the entries. For this reason $0 \leqslant s \leqslant i-1$ and $0 \leqslant t \leqslant N-1-i$.

(2) Walks have to visit all sites between i and j inclusive. Therefore, if $j \geqslant i$ it is $t \geqslant j-i$; if $j \leqslant i$ it is $s \geqslant i-j$. If $i = j$ both s and t cannot be zero.

(3) The sum on $n_{-s+1} \ldots n_t$ has the restriction that, with $u = m_s + \cdots + n_t$ the number of upward moves, the constraint $2u = j - i + p$ must be satisfied. (Note that for a bidiagonal matrix $[M^p]_{i,j} = 0$ if p and $|j-i|$ have opposite parities.)

The general expression for the counting numbers has been given by Krattenthaler [19], including the case of the null step. We discuss here the case of paths that, starting from the origin, only visit sites $i > 0$.

Proposition 7. *The number of positive walks of length p that never return to the origin and with prescribed numbers of upward steps is*

$$N_{0,t}(p;\ 1, n_2, \ldots n_t) = \prod_{j=2}^{y} \binom{n_j + n_{j-1} - 2}{n_j - 1} N(n_y, \ldots, n_t) \quad (3.50)$$

where $y = 2u - p$ is the height of the walk and $u = 1 + n_2 + \cdots + n_t$ is the total number of upward steps (see comment in proposition 5).

To obtain this counting number it is convenient to generalize the planar rooted tree by using edges of two types, with full or dotted lines, as in figure 6. There is a one-to-one map between the 'open mountain range' and the corresponding generalized planar rooted tree: to any full line edge of the tree correspond, as before, a pair of matched up-and-down sides of the mountain range, while a dotted line of the tree corresponds to a up side of the mountain range without the matching down side.

The correspondence between the number of visits at each level and the number of up steps is

$$\begin{aligned} N_j &= n_j + n_{j+1} - 1 \qquad 1 \leqslant j \leqslant y \\ N_j &= n_j + n_{j+1} \qquad y < j \leqslant t. \end{aligned} \quad (3.51)$$

Each vertex of the planar rooted tree is connected to a vertex at the lower level by a full line or a dotted line. n_j counts the full and dotted lines in the tree between level $j-1$ and level j; then

$$v_j = n_j \qquad j = 1, \ldots, t. \tag{3.52}$$

The number of planar rooted trees is evaluated first by drawing the root and a dotted path from the root to level y, with one vertex at each level $1, \ldots, y$, next by adding a number $\tilde{v}_j$ of new vertices at each level j, and connecting each vertex with a full line to a vertex at the lower level. If we call $\tilde{v}_j = v_j$ for $j = y+1, \ldots, t$ and $\tilde{v}_j = v_j - 1$ for $j = 1, \ldots, y$, we find the number of planar rooted trees with a given set of non-root vertices:

$$N(v_1, v_2, \ldots, v_t) = \binom{\tilde{v}_t + v_{t-1} - 1}{\tilde{v}_t}\binom{\tilde{v}_{t-1} + v_{t-2} - 1}{\tilde{v}_{t-1}} \ldots \binom{\tilde{v}_2 + v_1 - 1}{\tilde{v}_2}. \tag{3.53}$$

From this, equation (3.52) and the relation between $\tilde{v}_j$ and v_j we obtain the number in proposition 7.

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References

[1] Wigner E 1955 Characteristic vectors of bordered matrices with infinite dimensions: I *Ann. Math.* **62** 548–64

[2] Sinai Ya G and Soshnikov A B 1998 Central limit theorem for traces of large random symmetric matrices *Bol. Soc. Brazil. Mat.* **29** 1–24
Sinai Ya G and Soshnikov A B 1998 A refinement of Wigner's semicircle law in a neighborhood of the spectrum edge for random symmetric matrices *Funct. Anal. Appl.* **32** 114–31

[3] Hatano N and Nelson D R 1996 *Phys. Rev. Lett.* **77** 570
Hatano N and Nelson D R 1997 *Phys. Rev.* B **56** 8651

[4] Goldsheid I Y and Khoruzhenko B A 1998 Distribution of eigenvalues in non-Hermitian Anderson models *Phys. Rev. Lett.* **80** 2897
Goldsheid I Y and Khoruzhenko B A 2000 Eigenvalue curves of asymmetric tridiagonal random matrices *Electron. J. Prob.* **5** 1

[5] Feinberg J and Zee A 1999 Non-Hermitean localization and de-localization *Phys. Rev.* E **59** 6433–43

[6] Brezin E and Zee A 1998 Non-Hermitean delocalization: multiple scattering and bounds *Nucl. Phys.* B **509** 599–614

[7] Mudry C *et al* 1998 Density of states in the non-Hermitian Lloyd model *Phys. Rev.* B **58** 13 539–43

[8] Derrida B, Jacobsen J L and Zeitak R 2000 Lyapounov exponent and density of states of a one-dimensional non-Hermitian Schrodinger equation *J. Stat. Phys.* **98** 31–55

[9] Noh J D, Park H and den Nijs M 2000 *Phys. Rev. Lett.* **84** 3891
Noh J D, Park H, Kim D and den Nijs M 2001 *Phys. Rev.* E **64** 046131 (cond-mat/0103549)
Lee D S and den Nijs M 2002 *Phys. Rev.* E at press (cond-mat/0110485)

[10] Bauer M, Bernard D and Luck J M 2001 Even-visiting random walks: exact and asymptotic results in one dimension *J. Phys. A: Math. Gen.* **34** 2659 (cond-mat/0102512)

[11] Cicuta G M, Contedini M and Molinari L 2000 Non-Hermitian tridiagonal random matrices and returns to the origin of a random walk *J. Stat.Phys.* **98** 685–99

[12] Deutsch E 1999 Dyck path enumeration *Discrete Math.* **204** 167–202

[13] Goulden I P and Jackson D M 1983 *Combinatorial Enumeration* (New York: Wiley)

[14] Stanley R P 1999 *Enumerative Combinatorics* vol 1 (Cambridge: Cambridge University Press)

[15] Rubin R J and Weiss G H 1982 Random walks on lattices. The problem of visit to a set of points revisited *J. Math. Phys.* **23** 250–3

[16] Redner S and Kang K 1983 Asymptotic solution of interacting walks in one dimension *Phys. Rev. Lett.* **51** 1729–32

Redner S and Kang K 1984 Unimolecular reaction kinetics *Phys. Rev.* A **30** 3362–5

[17] Raykin M 1993 First-passage probability of a random walk on a disordered one-dimensional lattice *J. Phys. A: Math. Gen.* **26** 449–66

[18] Klarner D A 1970 Correspondence between plane trees and binary sequences *J. Comb. Theory* **9** 401–11

[19] Krattenthaler C 2000 Permutations with restricted patterns and Dyck paths *Preprint* math.Co/0002200

[20] Bernhart F R 1999 Catalan, Motzkin and Riordan numbers *Discrete Math.* **204** 73–112

[21] Stanley R P 1999 *Enumerative Combinatorics* vol 2 (Cambridge: Cambridge University Press)

[22] Mohanty G 1979 *Lattice Path Counting and Applications* (New York: Academic)

[23] Jonsson T and Wheater J F 1998 Area distribution for directed random walks *J. Stat. Phys.* **92** 713–25

[24] Merlini D, Sprugnoli R and Verri M C 1996 The area determined by under-diagonal lattice paths *Proc. CAAP (Lecture Notes in Computer Science vol 1059)* pp 59–71

[25] Chapman R 1999 Moments of Dyck paths *Discrete Math.* **204** 113–17

[26] Kirschenhofer P and Prodinger H 1996 Return statistics of simple random walks *J. Stat. Plan. Inference* **54** 67–74

[27] Gessel I, Goddard W, Shur W, Wilf H S and Yen L 1996 Counting pairs of lattice paths by intersections *J. Comb. Theory* A **74** 173–87 (math/9409212)

[28] Sachov V 1997 Probabilistic methods in combinatorial analysis *Encyclopedia of Mathematics and Its Applications* vol 56 (Cambridge: Cambridge University Press) ch 3

[29] Flajolet P and Guillemin F 1999 The formal theory of birth- and death processes, lattice path combinatorics and continued fractions *INRIA Report* no 3667

[30] Mansour T and Vainshtein A 2000 Restricted permutations, continued fractions and Chebyshev polynomials *Electron. J. Combin.* **7** 1–9

[31] Jani M and Rieper R G 2000 Continued fractions and Catalan problems *Preprint* math.CO/0001091

[32] Flajolet P 1980 Combinatorial aspects of continued fractions *Discrete Math.* **32** 125–61